1 Anisotropic Media

For isotropic media how the displacement field $\vec{D}$ is related to the electric field $\vec{E}$ and how the magnetic flux density $\vec{B}$ is related to the magnetic field strength $\vec{H}$ is described by the permittivity ε and the permeability μ.

$$\vec{E} = \varepsilon(\omega)\vec{E}, \quad \vec{B} = \mu(\omega) \tag{1.1}$$

For anisotropic media these relationships are instead described by tensors:

$$D_i = \varepsilon_{ij}(\omega)E_j, \quad B_i = \mu_{ij}(\omega)H_j \tag{1.2}$$

2 Plane Waves

Consider a plane wave in a media which is non-magnetic and transparent.

$$\boxed{D_i = \varepsilon_{ij}E_j, \quad \vec{B} = \vec{H}} \tag{2.1}$$

Since it is transparent, the components of the dielectric tensor are all real. Let's write down Maxwell's equations:

$$\nabla \times \vec{H} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t}, \quad \nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t} \tag{2.2}$$

Evaluating the time derivatives for $D, H \propto e^{-i\omega t}$,

$$\nabla \times \vec{H} = -\frac{i\omega}{c}\vec{D}, \quad \nabla \times \vec{E} = \frac{i\omega}{c}\vec{H} \tag{2.3}$$

Evaluating the curls for $\vec{E}, \vec{H} \propto e^{i\vec{k}\vec{r}}$,

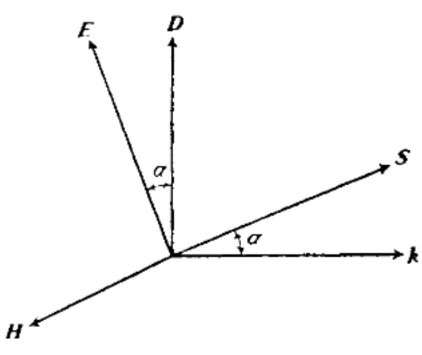
$$i\vec{k} \times \vec{H} = \frac{i\omega}{c}\vec{D}, \quad i\vec{k} \times \vec{E} = \frac{i\omega}{c}\vec{H} \tag{2.4}$$

Of course the i cancels out from both sides,

$$\boxed{\vec{k} \times \vec{H} = \frac{\omega}{c}\vec{D}, \quad \vec{k} \times \vec{E} = \frac{\omega}{c}\vec{H}} \tag{2.5}$$

We can already learn something about anisotropic media. From these two relations: the vectors $\vec{k}, \vec{H}, \vec{D}$ are perpendicular to each other and the three vectors $\vec{k}, \vec{E}, \vec{H}$ are perpendicular to each other. Remember $\vec{H} = \vec{B}$ so we can draw the following picture:

$$\vec{S} = \frac{c}{4\pi}\vec{E} \times \vec{H} \tag{2.6}$$



3 Wave-vector Surface

Now we can define the vector $\vec{n}$:

$$\vec{n} = \frac{c}{\omega}\vec{k} \tag{3.1}$$

Plugging this back into 2.5,

$$\boxed{\vec{H} = \vec{n} \times \vec{E}, \quad \vec{D} = -\vec{n} \times \vec{H}} \tag{3.2}$$

And the Poynting vector is

$$\vec{S} = \frac{c}{4\pi}\vec{E} \times \vec{H} = \frac{c}{4\pi}\vec{E} \times (\vec{n} \times \vec{E}) \tag{3.3}$$

$$\vec{S} = \frac{c}{4\pi}\left(E^2\vec{n} - (\vec{E} \cdot \vec{n})\vec{E}\right) \tag{3.4}$$

Combing the two equations in 3.2,

$$\boxed{\vec{D} = -\vec{n} \times (\vec{n} \times \vec{E}) = n^2\vec{E} - (\vec{n} \cdot \vec{E})\vec{n}} \tag{3.5}$$

Applying 2.1 to the left hand side,

$$\varepsilon\vec{E} = n^2\vec{E} - (\vec{n} \cdot \vec{E})\vec{n} \tag{3.6}$$

$$\vec{0} = n^2\vec{E} - (\vec{n} \cdot \vec{E})\vec{n} - \varepsilon\vec{E} \tag{3.7}$$

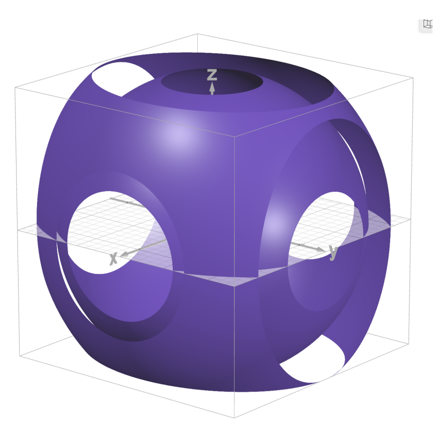
$$\boxed{\vec{0} = (n^2 I - \vec{n}\vec{n}^T - \varepsilon)\vec{E}} \tag{3.8}$$

To find the compatibility condition we will rotate our coordinate system to align with the right-handed orthonormal eigenbasis of ε. Let x, y, z be the axes of this eigenbasis. In this basis the matrix ε is diagonal with diagonal elements $\varepsilon_x, \varepsilon_y, \varepsilon_z$.

$$\vec{0} = \begin{pmatrix} n^2 - n_x^2 - \varepsilon_x & n_x n_y & n_x n_z \\ n_x n_y & n^2 - n_y^2 - \varepsilon_y & n_y n_z \\ n_x n_z & n_y n_z & n^2 - n_z^2 - \varepsilon_z \end{pmatrix} \vec{E} \tag{3.9}$$

Since this expression must hold for all $\vec{E}$, the determinant is zero. This produces the **Fresnel equation**:

$$\begin{aligned} &n^2(\varepsilon_x n_x^2 + \varepsilon_y n_y^2 + \varepsilon_z n_z^2) \\ &- \left[n_x^2\varepsilon_x(\varepsilon_y + \varepsilon_z) + n_y^2\varepsilon_y(\varepsilon_x + \varepsilon_z) + n_z^2\varepsilon_z(\varepsilon_x + \varepsilon_y)\right] \\ &+\varepsilon_x\varepsilon_y\varepsilon_z = 0 \end{aligned} \tag{3.10}$$



This equation can be solved for a particular frequency ω to give the magnitude of the vector $\vec{n}$ as a function of it's direction. This is a quadratic equation for n^2 so in general there are two different magnitudes $\vec{n}$ for a given direction. The function between ω and $\vec{k}$.

4 Ray surface

Another important vector to consider is the **ray vector**,

$$\vec{n} \cdot \vec{s} = 1 \tag{4.1}$$

Where the direction of $\vec{s}$ is determined by the group velocity vector $\partial\omega/\partial\vec{k}$. This vector is significant because the phase of light propagating from a point is

$$\Phi = \int \vec{n} \cdot d\vec{\ell} = \int (\vec{n} \cdot \vec{s}/s) d\ell = \int d\ell/s \tag{4.2}$$

In homogeneous media (where $\vec{n}$ doesn't depend on position) the phase is simply

$$\Phi = L/s \tag{4.3}$$

So the surface produced by the ray vector is such that the phase is the same at every point. This is the **ray surface**. The ray surface and the wave-vector surface are related. If the wave-vector surface is $f(\omega, \vec{k}) = 0$, then the group velocity vector is

$$\frac{\partial\omega}{\partial\vec{k}} = -\frac{\partial f/\partial\vec{k}}{\partial f/\partial\omega} \tag{4.4}$$

That is the group velocity is proportional to $\frac{\partial f}{\partial \vec{k}}$. But since $\vec{k} \propto \vec{n}$, the group velocity is also proportional to $\frac{\partial f}{\partial \vec{n}}$ which is the normal of the surface. So the ray vector is normal to the wave-vector surface. The reverse must also be true. So the wave vector is normal to the ray surface.

We can also prove that the ray vector must be in the same direction as the Poynting vector, as we would expect it to be. Starting from equation 3.2, we differentiate

$$\delta\vec{D} = \delta\vec{H} \times \vec{n} + \vec{H} \times \delta\vec{n} \tag{4.5}$$

$$\delta\vec{H} = \vec{n} \times \delta\vec{E} + \delta\vec{n} \times \vec{E} \tag{4.6}$$

Dotting both sides by $\vec{E}$ and $\vec{H}$ respectively,

$$\vec{E} \cdot \delta\vec{D} = \vec{H} \cdot \delta\vec{H} + \vec{E} \times \vec{H} \cdot \delta\vec{n} \tag{4.7}$$

$$\vec{H} \cdot \delta\vec{H} = \vec{D} \cdot \delta\vec{E} + \vec{E} \times \vec{H} \cdot \delta\vec{n} \tag{4.8}$$

Plugging the second equation into the first:

$$\vec{E} \cdot \delta\vec{D} = \vec{D} \cdot \delta\vec{E} + 2\vec{E} \times \vec{H} \cdot \delta\vec{n} \tag{4.9}$$

Since 2.1 is a linear relation, $\vec{D} \cdot \delta\vec{E} = \vec{E} \cdot \delta\vec{D}$.

$$\vec{E} \times \vec{H} \cdot \delta\vec{n} = 0 \tag{4.10}$$

Therefore, we get the expected result and the Poynting vector is normal to the wave-vectors surface and hence normal to s. Since the poynting vector is perpendicular to $\vec{H}$ and $\vec{E}$ the same is true for $\vec{s}$,

$$\vec{s} \cdot \vec{H} = 0, \qquad , \vec{s} \cdot \vec{E} = 0 \tag{4.11}$$

Recall equation 3.2,

$$s \times H = s \times (n \times \vec{E}) = \vec{n}(\vec{s} \cdot \vec{E}) - \vec{E}(\vec{n} \cdot \vec{s}) = -\vec{E} \tag{4.12}$$

$$s \times D = s \times (-n \times \vec{H}) = \vec{H}(\vec{n} \cdot \vec{s}) - \vec{n}(\vec{s} \cdot \vec{H}) = \vec{H} \tag{4.13}$$

$$\boxed{\vec{H} = \vec{s} \times \vec{D}, \quad \vec{s} \cdot \vec{E} = 0} \tag{4.14}$$

You may immediately recognize that this is very similar 3.2, and we can used this fact to immediately write the analogous expression:

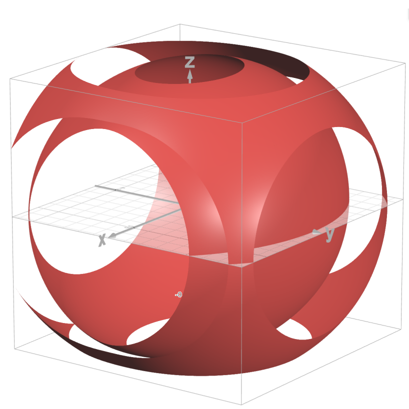
$$\vec{E} \mapsto \vec{D}, \quad \vec{n} \mapsto \vec{s}, \quad \varepsilon \mapsto \varepsilon^{-1} \tag{4.15}$$

$$\boxed{\vec{0} = (s^2 I - \vec{s}\vec{s}^T - \varepsilon^{-1})\vec{D}} \tag{4.16}$$

Just as before the determinant must be 0,

$$\begin{aligned} s^2(\varepsilon_y\varepsilon_z s_x^2 + \varepsilon_x\varepsilon_z s_y^2 + \varepsilon_x\varepsilon_y s_z^2) \\ - \left[s_x^2(\varepsilon_y + \varepsilon_z) + s_y^2(\varepsilon_x + \varepsilon_z) + s_z^2(\varepsilon_x + \varepsilon_y)\right] + 1 = 0 \end{aligned} \tag{4.17}$$

Just like the fresnel equation this equation results in two solution for a given direction s.



5 Polarization

To find the polarization, consider when $\vec{n}$ is perpendicular to $\vec{E}$, from equation 3.5 for D_α in the transverse direction,

$$D_\alpha = n^2 E_\alpha \tag{5.1}$$

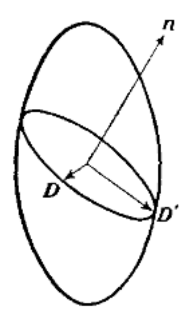
Substituting, $E_\alpha = \varepsilon^{-1}_{\alpha\beta} D_\beta$ we have

$$\left(\frac{1}{n^2}\delta_{\alpha\beta} - \varepsilon^{-1}_{\alpha\beta}\right) D_\beta = 0 \tag{5.2}$$

Just as before the determinant must be zero,

$$\frac{x^2}{\varepsilon_x} + \frac{y^2}{\varepsilon_y} + \frac{z^2}{\varepsilon_z} = 1 \tag{5.3}$$

This describes the two axis of polarization in each direction,



A similar construction can be written for the electric field

$$\varepsilon_x x^2 + \varepsilon_y y^2 + \varepsilon_z z^2 = 1 \tag{5.4}$$

6 Uniaxial Crystals

The optical properties of crystals generally fall into three categories, cubic, uniaxial and biaxial. For a uniaxial crystal, two of the eigenvalues of the permittivity are equal. In fresnels equations we let $\varepsilon_x = \varepsilon_y = \varepsilon_\perp$ and $\varepsilon_z = \varepsilon_\parallel$,

$$(n^2 - \varepsilon_\perp)\left[\varepsilon_\parallel n_z^2 + \varepsilon_\perp(n_x^2 + n_y^2) - \varepsilon_\perp\varepsilon_\parallel\right] = 0 \tag{6.1}$$

This is simply a sphere and an ellipse:

$$n^2 = \varepsilon_\perp \tag{6.2}$$

$$\frac{n_z^2}{\varepsilon_\perp} + \frac{n_x^2 + n_y^2}{\varepsilon_\parallel} = 1 \tag{6.3}$$

Similarly for the ray surface:

$$s^2 = \frac{1}{\varepsilon_\perp} \tag{6.4}$$

$$\varepsilon_\perp s_z^2 + \varepsilon_\parallel(s_x^2 + s_y^2) = 1 \tag{6.5}$$