Problem set 2 — Solutions

Phy-801

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1. **Drude dynamics and the single relaxation time approximation**

   (a) In a short time interval dt:

   - with probability $1 - dt/\tau$ there is no collision and

   $$\boldsymbol{p}(t+dt) = \boldsymbol{p}(t) - e\boldsymbol{E}\,dt, \tag{1}$$

   - with probability dt/τ there is a collision and $\boldsymbol{p}(t+dt) = \boldsymbol{0}$.

   Taking the ensemble average,

   $$\begin{aligned}\langle \boldsymbol{p}(t+dt)\rangle &= \left(1 - \frac{dt}{\tau}\right)\big(\langle \boldsymbol{p}(t)\rangle - e\boldsymbol{E}\,dt\big) + \frac{dt}{\tau}\boldsymbol{0} \\ &= \langle \boldsymbol{p}(t)\rangle - e\boldsymbol{E}\,dt - \frac{dt}{\tau}\langle \boldsymbol{p}(t)\rangle + O(dt^2).\end{aligned} \tag{2}$$

   In addition

   $$\langle \boldsymbol{p}(t+dt)\rangle = \langle \boldsymbol{p}(t)\rangle + \frac{d\langle \boldsymbol{p}\rangle}{dt}dt \tag{3}$$

   Comparing these equations, and keeping only terms to leading order in dt we obtain

   $$\boxed{\frac{d\langle \boldsymbol{p}\rangle}{dt} = -e\boldsymbol{E} - \frac{\langle \boldsymbol{p}\rangle}{\tau}} \tag{4}$$

   as required.

   (b) In steady state, $d\langle \boldsymbol{p}\rangle/dt = 0$, so from (4)

   $$0 = -e\boldsymbol{E} - \frac{\boldsymbol{p}_\mathrm{d}}{\tau} \quad \Rightarrow \quad \boldsymbol{p}_\mathrm{d} = -e\tau\,\boldsymbol{E}. \tag{5}$$

   The drift velocity is

   $$\boldsymbol{v}_\mathrm{d} = \frac{\boldsymbol{p}_\mathrm{d}}{m} = -\frac{e\tau}{m}\boldsymbol{E}. \tag{6}$$

   The current density is

   $$\boldsymbol{j} = -ne\boldsymbol{v}_\mathrm{d} = -ne\left(-\frac{e\tau}{m}\boldsymbol{E}\right) = \frac{ne^2\tau}{m}\boldsymbol{E}. \tag{7}$$

   Thus Ohm's law $\boldsymbol{j} = \sigma_0\boldsymbol{E}$ gives

   $$\boxed{\sigma_0 = \frac{ne^2\tau}{m}} \tag{8}$$

   the Drude conductivity.

   (c) Consider the s dependency for fixed t and $s > 0$. By direct generalisation of the approach above

   $$\langle p_i(t+s+ds)p_j(t)\rangle = \left(1 - \frac{ds}{\tau}\right)\langle (p_i(t+s) - eE_i ds)p_j(t)\rangle + \left(\frac{ds}{\tau}\right)\langle 0\rangle \tag{9}$$

Expand both sides to linear order in ds and simplify

$$\frac{d}{ds}\langle p_i(t+s)p_j(t)\rangle = -\frac{1}{\tau}\langle (p_i(t+s))p_j(t)\rangle - eE_i\langle p_j(t)\rangle \tag{10}$$

using $C_{ij} = \langle p_i(t+s)p_j(t)\rangle - \langle p_i(t+s)\rangle\langle p_j(t)\rangle$ and $\langle p_j(t)\rangle = -e\tau E_j$

$$\frac{dC_{ij}}{ds} = -\frac{C_{ij}}{\tau} \qquad \Longrightarrow \qquad C_{ij}(s,t) = C_{ij}(0,t)e^{-s/\tau}, \quad s>0. \tag{11}$$

Note

- C_{ij} is t-independent $C_{ij}(s,t) = C_{ij}(s,0)$
- by its definition $C_{ij}(s,t) = C_{ji}(-s,t+s)$

taken together

$$C_{ij}(s,t) = C_{ij}(s,0) = C_{ij}(-s,0) \tag{12}$$

from which it follows

$$\boxed{C_{ij}(s,t) = C_{ij}(0,0)e^{-|s|/\tau}} \tag{13}$$

(d) Use

$$N = \int d^3\boldsymbol{r}\, d^3\boldsymbol{p}\, f(\boldsymbol{r},\boldsymbol{p},t), \tag{14}$$

and

$$\langle \boldsymbol{p}\rangle = \frac{1}{N}\int d^3\boldsymbol{r}\, d^3\boldsymbol{p}\, f(\boldsymbol{r},\boldsymbol{p},t)\,\boldsymbol{p}. \tag{15}$$

Differentiating,

$$\frac{d\langle p_i\rangle}{dt} = \frac{1}{N}\int d^3\boldsymbol{r}\, d^3\boldsymbol{p}\, p_i\,\frac{\partial f}{\partial t}. \tag{16}$$

Using the equation of motion for f

$$\frac{d\langle p_i\rangle}{dt} = \frac{1}{N}\int d^3\boldsymbol{r}\, d^3\boldsymbol{p}\, p_i\left[e\boldsymbol{E}\cdot\nabla_{\boldsymbol{p}} f + \int d^3\boldsymbol{q}\big(W_{\boldsymbol{q}\to\boldsymbol{p}} f(\boldsymbol{q}) - W_{\boldsymbol{p}\to\boldsymbol{q}} f(\boldsymbol{p})\big)\right]. \tag{17}$$

For the field term, integrate by parts in $\boldsymbol{p}$ (assuming $f\to 0$ as $|\boldsymbol{p}|\to\infty$):

$$\begin{aligned}\int d^3\boldsymbol{p}\, p_i\,\boldsymbol{E}\cdot\nabla_{\boldsymbol{p}} f &= E_j\int d^3\boldsymbol{p}\, p_i\frac{\partial f}{\partial p_j}\\ &= -E_j\int d^3\boldsymbol{p}\,\delta_{ij} f = -E_i\int d^3\boldsymbol{p}\, f.\end{aligned} \tag{18}$$

Including $\boldsymbol{r}$ gives $-E_iN$, and therefore

$$\frac{1}{N}\int d^3\boldsymbol{r}\, d^3\boldsymbol{p}\, p_i\, e\boldsymbol{E}\cdot\nabla_{\boldsymbol{p}} f = -eE_i. \tag{19}$$

For the collision term, define

$$I_i = \frac{1}{N}\int d^3\boldsymbol{r}\, d^3\boldsymbol{p}\, p_i\int d^3\boldsymbol{q}\,\big(W_{\boldsymbol{q}\to\boldsymbol{p}} f(\boldsymbol{q}) - W_{\boldsymbol{p}\to\boldsymbol{q}} f(\boldsymbol{p})\big). \tag{20}$$

Exchange $\boldsymbol{p}\leftrightarrow\boldsymbol{q}$ in the first term:

$$\begin{aligned}I_i &= \frac{1}{N}\int d^3\boldsymbol{r}\int d^3\boldsymbol{p}\int d^3\boldsymbol{q}\,[q_iW_{\boldsymbol{p}\to\boldsymbol{q}} f(\boldsymbol{p}) - p_iW_{\boldsymbol{p}\to\boldsymbol{q}} f(\boldsymbol{p})]\\ &= \frac{1}{N}\int d^3\boldsymbol{r}\int d^3\boldsymbol{p}\, f(\boldsymbol{p})\left[\int d^3\boldsymbol{q}\,(q_i-p_i)W_{\boldsymbol{p}\to\boldsymbol{q}}\right].\end{aligned} \tag{21}$$

Hence

$$\frac{d\langle p_i\rangle}{dt} = -eE_i + \frac{1}{N}\int d^3\boldsymbol{r}\int d^3\boldsymbol{p}\, f(\boldsymbol{p})\left[\int d^3\boldsymbol{q}\,(q_i - p_i)W_{\boldsymbol{p}\to\boldsymbol{q}}\right]. \tag{22}$$

Hence using the condition in the question, the bracket equals $-p_i/\tau$, so

$$\begin{aligned}\frac{d\langle p_i\rangle}{dt} &= -eE_i - \frac{1}{N}\int d^3\boldsymbol{r}\int d^3\boldsymbol{p}\, f(\boldsymbol{p})\frac{p_i}{\tau}\\ &= -eE_i - \frac{\langle p_i\rangle}{\tau}.\end{aligned} \tag{23}$$

Therefore

$$\boxed{\frac{d\langle\boldsymbol{p}\rangle}{dt} = -e\boldsymbol{E} - \frac{\langle\boldsymbol{p}\rangle}{\tau}} \tag{24}$$

(e) By direct substitution

$$\begin{aligned}\int d^3\boldsymbol{q}\,(\boldsymbol{q}-\boldsymbol{p})W_{\boldsymbol{p}\to\boldsymbol{q}} &= \frac{1}{n\tau}\int d^3\boldsymbol{q}\,(\boldsymbol{q}-\boldsymbol{p})f_0(\boldsymbol{q})\\ &= \frac{1}{n\tau}\left[\int d^3\boldsymbol{q}\,\boldsymbol{q}f_0(\boldsymbol{q}) - \boldsymbol{p}\int d^3\boldsymbol{q}\, f_0(\boldsymbol{q})\right].\end{aligned} \tag{25}$$

Using the conditions

$$\int d^3\boldsymbol{q}\, f_0(\boldsymbol{q})\,\boldsymbol{q} = \boldsymbol{0}, \qquad \int d^3\boldsymbol{q}\, f_0(\boldsymbol{q}) = n, \tag{26}$$

we obtain

$$\int d^3\boldsymbol{q}\,(\boldsymbol{q}-\boldsymbol{p})W_{\boldsymbol{p}\to\boldsymbol{q}} = -\frac{\boldsymbol{p}}{\tau}, \tag{27}$$

Next, compute the collision integral:

$$I_{\text{coll}}(\boldsymbol{r},\boldsymbol{p},t) = \int d^3\boldsymbol{q}\,[W_{\boldsymbol{q}\to\boldsymbol{p}}f(\boldsymbol{r},\boldsymbol{q},t) - W_{\boldsymbol{p}\to\boldsymbol{q}}f(\boldsymbol{r},\boldsymbol{p},t)] \tag{28}$$

$$\begin{aligned}&= \int d^3\boldsymbol{q}\left[\frac{f_0(\boldsymbol{p})}{n\tau}f(\boldsymbol{q}) - \frac{f_0(\boldsymbol{q})}{n\tau}f(\boldsymbol{p})\right]\\ &= \frac{1}{n\tau}\left[f_0(\boldsymbol{p})\int d^3\boldsymbol{q}\, f(\boldsymbol{q}) - f(\boldsymbol{p})\int d^3\boldsymbol{q}\, f_0(\boldsymbol{q})\right]\\ &= \frac{1}{n\tau}\left[f_0(\boldsymbol{p})\,n - f(\boldsymbol{p})\,n\right]\\ &= \frac{f_0(\boldsymbol{p}) - f(\boldsymbol{p})}{\tau}.\end{aligned} \tag{29}$$

Hence we obtain

$$\boxed{\frac{\partial f}{\partial t} - e\boldsymbol{E}\cdot\nabla_{\boldsymbol{p}}f = \frac{f_0 - f}{\tau}} \tag{30}$$

2. **A failure of Drude's theory: the thermoelectric correction**

(a) For a static solution, $\partial f/\partial t = 0$, so the equation of motion becomes

$$\dot{\boldsymbol{r}}\cdot\nabla_{\boldsymbol{r}}f + \dot{\boldsymbol{p}}\cdot\nabla_{\boldsymbol{p}}f = \frac{f_0 - f}{\tau}. \tag{31}$$

Substitute $f = f_0 + \delta f_1 + \delta f_2$, keep only terms linear in $\boldsymbol{X}_j$, and treat the responses to $\boldsymbol{X}_1, \boldsymbol{X}_2$ separately.

Response to $\boldsymbol{\mathcal{E}}$: when T is uniform, $\nabla_{\boldsymbol{r}}f_0 = 0$ and $\dot{\boldsymbol{p}} = -e\boldsymbol{\mathcal{E}}$. To linear order,

$$\dot{\boldsymbol{r}}\cdot\nabla_{\boldsymbol{r}}\delta f_1 + \dot{\boldsymbol{p}}\cdot\nabla_{\boldsymbol{p}}f_0 = -\frac{\delta f_1}{\tau}. \tag{32}$$

Dropping $\nabla_{\boldsymbol{r}}\delta f_1$ (second order in gradients) and using $\dot{\boldsymbol{p}} = -e\boldsymbol{\mathcal{E}}$,

$$-e\boldsymbol{\mathcal{E}} \cdot \nabla_{\boldsymbol{p}} f_0 = -\frac{\delta f_1}{\tau} \quad \Rightarrow \quad \delta f_1 = \tau e \boldsymbol{\mathcal{E}} \cdot \nabla_{\boldsymbol{p}} f_0. \tag{33}$$

For Maxwell–Boltzmann,

$$\nabla_{\boldsymbol{p}} f_0 = -\frac{\boldsymbol{p}}{m k_B T} f_0 = -\frac{\boldsymbol{v}}{k_B T} f_0, \tag{34}$$

so

$$\boxed{\delta f_1 = -\frac{\tau e}{k_B T}\, \boldsymbol{\mathcal{E}} \cdot \boldsymbol{v}\, f_0 = \frac{\boldsymbol{F}_1 \cdot \boldsymbol{v}\, \tau}{k_B T} f_0, \qquad \boldsymbol{F}_1 = -e\boldsymbol{\mathcal{E}}.} \tag{35}$$

Response to ∇T: now set $\boldsymbol{\mathcal{E}} = 0$, so $\dot{\boldsymbol{p}} = 0$ and only the spatial variation of f_0 contributes:

$$\dot{\boldsymbol{r}} \cdot \nabla_{\boldsymbol{r}} f_0 = -\frac{\delta f_2}{\tau} \quad \Rightarrow \quad \delta f_2 = -\tau\, \boldsymbol{v} \cdot \nabla_{\boldsymbol{r}} f_0. \tag{36}$$

With f_0 depending on $\boldsymbol{r}$ only through $T(\boldsymbol{r})$,

$$\nabla_{\boldsymbol{r}} f_0 = \frac{\partial f_0}{\partial T}\, \nabla T = f_0 \frac{\partial \log f_0}{\partial T}\, \nabla T \tag{37}$$

where by direct calculation

$$\frac{\partial \log f_0}{\partial T} = -\frac{3}{2T} + \frac{p^2}{2 m k_B T^2} \tag{38}$$

Therefore

$$\delta f_2 = -\tau f_0\, \frac{\boldsymbol{v} \cdot \nabla T}{k_B T^2} \left(\frac{p^2}{2m} - \frac{3}{2} k_B T \right). \tag{39}$$

Writing this as

$$\delta f_2 = \frac{\boldsymbol{F}_2 \cdot \boldsymbol{v}\, \tau}{k_B T} f_0 \tag{40}$$

gives

$$\boxed{\boldsymbol{F}_2 = -\left(\frac{p^2}{2m} - \frac{3}{2} k_B T \right) \frac{\nabla T}{T}} \tag{41}$$

as required.

(b) To linear order,

$$f = f_0 + \delta f_1 + \delta f_2. \tag{42}$$

The equilibrium contribution integrates to zero so

$$\langle \boldsymbol{j} \rangle = \int d^3\boldsymbol{p}\, (-e\boldsymbol{v})(\delta f_1 + \delta f_2), \tag{43}$$

and L_{12} comes from the δf_2 term (proportional to $-\nabla T$). Using

$$\delta f_2 = \frac{\boldsymbol{F}_2 \cdot \boldsymbol{v}\, \tau}{k_B T} f_0, \qquad \boldsymbol{F}_2 = -\left(\varepsilon - \langle \varepsilon \rangle\right) \frac{\nabla T}{T}, \tag{44}$$

we find

$$\begin{aligned} \langle j_i \rangle &= -e \int d^3 p v_i\, \delta f_2 \\ &= -\frac{e\tau}{k_B T} \int d^3 p v_i \boldsymbol{F}_2 \cdot \boldsymbol{v} f_0, \\ &= -\frac{e\tau n}{k_B T} \langle v_i \boldsymbol{F}_2 \cdot \boldsymbol{v} \rangle \end{aligned} \tag{45}$$

we then evaluate the equilibrium expectation value

$$\begin{aligned}
\langle v_i \boldsymbol{F}_2 \cdot \boldsymbol{v} \rangle &= -\langle v_i v_j (\varepsilon - \langle \varepsilon \rangle) \rangle \frac{\partial_j T}{T} \\
&= -\tfrac{1}{3} \delta_{ij} \left\langle v^2 (\varepsilon - \langle \varepsilon \rangle) \right\rangle \frac{\partial_j T}{T} \\
&= -\frac{2}{3m} \langle \varepsilon (\varepsilon - \langle \varepsilon \rangle) \rangle \frac{\partial_i T}{T} \\
&= -\frac{2}{3m} \sigma_\varepsilon^2 \frac{\partial_i T}{T} \\
&= -\frac{1}{m} (k_{\mathrm{B}} T)^2 \frac{\partial_i T}{T}
\end{aligned} \tag{46}$$

from which we get

$$\begin{aligned}
\langle j_i \rangle &= \frac{e\tau n}{k_{\mathrm{B}} T} \cdot \frac{1}{m} (k_{\mathrm{B}} T)^2 \cdot \frac{\partial_i T}{T} \\
&= \frac{e\tau n k_{\mathrm{B}}}{m} \partial_i T
\end{aligned} \tag{47}$$

and hence

$$\boxed{L_{12} = -\frac{e\tau n k_{\mathrm{B}}}{m}} \tag{48}$$

(c) From

$$\begin{pmatrix} \langle \boldsymbol{j} \rangle \\ \langle \boldsymbol{j}_q \rangle \end{pmatrix} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{\mathcal{E}} \\ -\nabla T \end{pmatrix}, \tag{49}$$

we identify:

- For $\nabla T = 0$, $\langle \boldsymbol{j} \rangle = L_{11} \boldsymbol{\mathcal{E}} = \sigma \boldsymbol{\mathcal{E}}$, so
$$L_{11} = \sigma. \tag{50}$$
- For open circuit $\langle \boldsymbol{j} \rangle = 0$,
$$0 = L_{11} \boldsymbol{\mathcal{E}} + L_{12}(-\nabla T) \quad \Rightarrow \quad \boldsymbol{\mathcal{E}} = \frac{L_{12}}{L_{11}} \nabla T. \tag{51}$$
Comparing with $\boldsymbol{\mathcal{E}} = S \nabla T$ gives
$$S = \frac{L_{12}}{L_{11}} \quad \Rightarrow \quad L_{12} = \sigma S. \tag{52}$$
- For $\nabla T = 0$, $\langle \boldsymbol{j} \rangle = L_{11} \boldsymbol{\mathcal{E}}$ and
$$\left\langle \boldsymbol{j}_q \right\rangle = L_{21} \boldsymbol{\mathcal{E}} = \Pi \langle \boldsymbol{j} \rangle = \Pi L_{11} \boldsymbol{\mathcal{E}}, \tag{53}$$
so
$$L_{21} = \Pi L_{11} = \Pi \sigma. \tag{54}$$
- For $\langle \boldsymbol{j} \rangle = 0$, $\boldsymbol{\mathcal{E}} = S \nabla T$ and
$$\left\langle \boldsymbol{j}_q \right\rangle = L_{21} \boldsymbol{\mathcal{E}} + L_{22}(-\nabla T) = L_{21} S \nabla T - L_{22} \nabla T. \tag{55}$$
as $\left\langle \boldsymbol{j}_q \right\rangle = -\kappa \nabla T$
$$-\kappa \nabla T = (L_{21} S - L_{22}) \nabla T \quad \Rightarrow \quad \kappa = L_{22} - S L_{21}. \tag{56}$$

Using the Kelvin relation $\Pi = TS$ and $L_{11} = \sigma$, we can write

$$L_{21} = \Pi \sigma = T S \sigma = T L_{12}. \tag{57}$$

Collecting these,

$$\boxed{\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} = \begin{pmatrix} \sigma & \sigma S \\ T \sigma S & \kappa + T \sigma S^2 \end{pmatrix}} \tag{58}$$

(d) From above,

$$S = \frac{L_{12}}{L_{11}}. \tag{59}$$

For Drude,

$$L_{11} = \sigma_0 = \frac{ne^2\tau}{m}, \qquad L_{12} = -\frac{en\tau k_B}{m}. \tag{60}$$

Therefore

$$\begin{aligned} S &= \frac{-en\tau k_B/m}{ne^2\tau/m} \\ &= -\frac{k_B}{e}. \end{aligned} \tag{61}$$

So

$$\boxed{S = -\frac{k_B}{e}} \tag{62}$$

(e) i. For $\boldsymbol{\mathcal{E}} = 0$

$$\langle \boldsymbol{j}_q \rangle = -L_{22}\nabla T = -(\kappa + \sigma S^2 T)\nabla T. \tag{63}$$

and hence

$$\boxed{\kappa_{\text{closed}} = \kappa + \sigma S^2 T} \tag{64}$$

ii. Using the result above,

$$\boxed{\delta\tilde{\kappa} = \frac{\kappa_{\text{closed}} - \kappa}{\kappa} = \frac{\sigma S^2 T}{\kappa}} \tag{65}$$

(f) From part (2(e)ii),

$$\delta\tilde{\kappa} = \frac{\sigma S^2 T}{\kappa}. \tag{66}$$

In Drude theory,

$$S = -k_B/e, \qquad \sigma_0 = \frac{ne^2\tau}{m}, \qquad \kappa = \frac{5}{2}\frac{n\tau k_B^2 T}{m} \tag{67}$$

combining these results we get

$$\boxed{\delta\tilde{\kappa} = \frac{\sigma S^2 T}{\kappa} = \frac{2}{5}} \tag{68}$$

For copper at room temperature, using

$$S \approx 2 \times 10^{-6}\,\text{V/K}, \quad \sigma_0 \approx 5.9 \times 10^{7}\,\Omega^{-1}\text{m}^{-1}, \quad \kappa \approx 400\,\text{W}\,\text{m}^{-1}\text{K}^{-1}, \tag{69}$$

and $T \approx 300\,\text{K}$, we estimate

$$\begin{aligned} \delta\tilde{\kappa}_{\text{Cu}} &= \frac{\sigma_0 S^2 T}{\kappa} \\ &\approx \frac{(5.9 \times 10^{7})(2 \times 10^{-6})^2(300)}{400} \\ &\approx 1.8 \times 10^{-4}. \end{aligned} \tag{70}$$

Thus

$$\boxed{\delta\tilde{\kappa}_{\text{Cu}} \sim 10^{-4} \ll \frac{2}{5},} \tag{71}$$

whereas Drude predicts an $O(1)$ correction. Drude theory therefore dramatically overestimates the thermoelectric correction to the thermal conductivity.

3. **The plasma frequency and the transparency of metals**

(a) Fourier transform of the Drude equation of motion gives

$$-i\omega \langle \boldsymbol{p}(\omega) \rangle = -e\boldsymbol{E}(\omega) - \frac{\langle \boldsymbol{p}(\omega) \rangle}{\tau}, \tag{72}$$

solving for $\langle \boldsymbol{p}(\omega) \rangle$

$$\langle \boldsymbol{p}(\omega) \rangle = -\frac{e\tau}{1 - i\omega\tau} \boldsymbol{E}(\omega). \tag{73}$$

The current density is $\boldsymbol{j} = -en \langle \boldsymbol{p} \rangle / m$, and hence

$$\boldsymbol{j}(\omega) = \frac{ne^2\tau}{m} \frac{1}{1 - i\omega\tau} \boldsymbol{E}(\omega). \tag{74}$$

Thus, by the definition of $\boldsymbol{j}(\omega) = \sigma(\omega)\boldsymbol{E}(\omega)$

$$\boxed{\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau}, \qquad \sigma_0 = \frac{ne^2\tau}{m}} \tag{75}$$

(b) Take the curl of Faraday's law:

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\frac{\partial}{\partial t}(\nabla \times \boldsymbol{B}). \tag{76}$$

Using Ampère's law, $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j} + \epsilon_0\mu_0 \frac{\partial \boldsymbol{E}}{\partial t}$, we get

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\mu_0 \frac{\partial \boldsymbol{j}}{\partial t} - \epsilon_0\mu_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2}. \tag{77}$$

Using $\nabla \times (\nabla \times \boldsymbol{E}) = \nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E}$ and $\nabla \cdot \boldsymbol{E} = 0$,

$$\nabla^2 \boldsymbol{E} = \mu_0 \frac{\partial \boldsymbol{j}}{\partial t} + \epsilon_0\mu_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2}, \tag{78}$$

and by Fourier transform

$$\nabla^2 \boldsymbol{E}(\omega) = -i\omega\mu_0 \boldsymbol{j}(\omega) - \epsilon_0\mu_0\omega^2 \boldsymbol{E}(\omega). \tag{79}$$

Using $c^{-2} = \epsilon_0\mu_0$, and $\boldsymbol{j}(\omega) = \sigma(\omega)\boldsymbol{E}(\omega)$

$$\nabla^2 \boldsymbol{E}(\omega) = -\frac{\omega^2}{c^2} \left[1 + i\frac{\sigma(\omega)}{\epsilon_0\omega}\right] \boldsymbol{E}(\omega). \tag{80}$$

hence we identify

$$\boxed{\epsilon(\omega) = 1 + i\frac{\sigma(\omega)}{\epsilon_0\omega}} \tag{81}$$

(c) For $\omega\tau \gg 1$

$$\sigma(\omega) = \frac{i\sigma_0}{\omega\tau} \left(1 + O(\omega\tau)^{-1}\right) \tag{82}$$

keeping only the leading order term

$$\epsilon(\omega) = 1 + i\frac{\sigma(\omega)}{\epsilon_0\omega} = 1 - \frac{\sigma_0}{\epsilon_0\omega^2\tau} \tag{83}$$

comparing with

$$\epsilon(\omega) = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2} \tag{84}$$

we identify

$$\boxed{\omega_{\mathrm{p}} = \sqrt{\frac{\sigma_0}{\epsilon_0\tau}} = \sqrt{\frac{ne^2}{m\epsilon_0}}} \tag{85}$$

(d) A decaying solution is of the form $\boldsymbol{E}(z,\omega) = \boldsymbol{E}_0(\omega)e^{-z/\xi}$ plugging this into

$$\nabla^2 \boldsymbol{E} = -\frac{\omega^2}{c^2}\epsilon(\omega)\boldsymbol{E} \tag{86}$$

we obtain

$$\frac{1}{\xi^2}\boldsymbol{E} = -\frac{\omega^2}{c^2}\left(1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2}\right)\boldsymbol{E} \tag{87}$$

solving for ξ

$$\boxed{\xi = \frac{c}{\sqrt{\omega_{\mathrm{p}}^2 - \omega^2}}} \tag{88}$$

which diverges as $\omega \to \omega_{\mathrm{p}}^-$.

(e) An oscillating solution is of the form $\boldsymbol{E}(z,\omega) = \boldsymbol{E}_0(\omega)e^{2\pi i z/\lambda}$ plugging this into (86) we obtain

$$-\frac{4\pi^2}{\lambda^2}\boldsymbol{E} = -\frac{\omega^2}{c^2}\left(1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2}\right)\boldsymbol{E} \tag{89}$$

solving for λ

$$\boxed{\lambda = \frac{2\pi c}{\sqrt{\omega^2 - \omega_{\mathrm{p}}^2}}} \tag{90}$$

which diverges as $\omega \to \omega_{\mathrm{p}}^+$

(f) For a monovalent metal, the conduction electron density equals the ionic number density:

$$n = \frac{N_{\mathrm{A}}\rho}{m_{\mathrm{mol}}}. \tag{91}$$

Using $\rho = 0.534\,\mathrm{g\,cm^{-3}}$, $m_{\mathrm{mol}} = 6.94\,\mathrm{g\,mol^{-1}}$, $N_{\mathrm{A}} \approx 6.02 \times 10^{23}\,\mathrm{mol^{-1}}$, we find

$$n \approx \frac{(6.02 \times 10^{23})(0.534)}{6.94}\,\mathrm{cm^{-3}} \approx 4.6 \times 10^{22}\,\mathrm{cm^{-3}} \approx 4.6 \times 10^{28}\,\mathrm{m^{-3}}. \tag{92}$$

The plasma frequency is

$$\omega_{\mathrm{p}} = \sqrt{\frac{ne^2}{m\epsilon_0}}, \tag{93}$$

where m is the electron mass. Plugging in $n \approx 4.6 \times 10^{28}\,\mathrm{m^{-3}}$, $e \approx 1.60 \times 10^{-19}\,\mathrm{C}$, $m \approx 9.11 \times 10^{-31}\,\mathrm{kg}$, and $\epsilon_0 \approx 8.85 \times 10^{-12}\,\mathrm{F\,m^{-1}}$ gives

$$\omega_{\mathrm{p}} \approx 1.2 \times 10^{16}\,\mathrm{s^{-1}}. \tag{94}$$

The plasma wavelength

$$\boxed{\lambda_{\mathrm{p}} = \frac{2\pi c}{\omega_{\mathrm{p}}} \approx \frac{2\pi(3.0 \times 10^8)}{1.2 \times 10^{16}}\,\mathrm{m} \approx 1.6 \times 10^{-7}\,\mathrm{m} \approx 1600\,\text{Å}} \tag{95}$$

Experimentally, lithium becomes transparent at about 1850 Å. The simple Drude estimate for λ_{p} is therefore within $\sim$ 15–20% of experiment, which is good given the simplicity of the model.